Galerkin Method and Nonlinear PDEs

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“Could a butterfly in a West Africa rain forest, by flitting to the left of a tree than to the right, possibly set into motion a chain of events that escalates into a hurricane striking coastal South Carolina a few weeks later? As bizarre as this premise may sound, research of the last few decades suggests that the answer is yes, and that the effect is hardly limited to butterflies ”

Page 263, “Perils of a Restless Planet”, Ernest Zebrowski, Jr.

1. Lord Rayleigh, 1916, "On convection currents in a horizontal layer of a fluid, when the higher temperature is on the under side", *Phil. Mag.*, (6), **32**, pp. 529–546.

2. Drazin, P. G., Hydrodynamic Stability, 1981, Cambridge Press, pp. 32 – 68.

3. Lorenz, E. N., 1963 "Deterministic nonperiodic flow", J. atmos. sci., **20**, pp. 130 – 141.

4. Saltzman, B. 1962, "Finite amplitude free convection as an initial value problem – I", J. atmos. sci., **19**, pp. 329 – 341.

5. Veronis, George, 1963, "An analysis of wind-driven ocean circulation with a limited number of Fourier components", J. atmos. sci., **20**, pp. 577 – 593.

1. MIDN Blair Bolt, ’00

2. MIDN James Coleman, ’99

3. MIDN Luke Whittemore, ’01

4. MIDN Ben Heineike, ’02

5. MIDN Pete Bosler, ’02

Veronis's Initial-Boundary Value Problem

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\frac{\partial\psi}{\partial x} = -K\Delta\psi + \frac{g}{H},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

Initial-Boundary Value Problem

Solve the PDE with

$$g = -\frac{W}{L}\sin\frac{x}{L}\sin\frac{y}{L}, \quad \text{wind stress}$$

$$\psi(x, y, 0) = \text{given}$$

and

$$\psi = 0, \quad \text{when} \quad x = y = 0, \quad \text{and} \quad x = y = \pi L.$$

Nondimensionalize variables:

$$x = Lx', \quad y = Ly' \quad t = \frac{t'}{L\beta}, \quad \psi = \frac{W}{\beta H}\psi'$$

New PDE:

$$\Delta\psi_t + RJ(\psi, \Delta\psi) + \frac{\partial\psi}{\partial x} = -\epsilon\Delta\psi - \sin x \sin y,$$

where

$$R = \frac{W}{\beta^2 H L^3}, \qquad \epsilon = \frac{K}{\beta L}.$$

Initial condition

$$\psi(x, y, 0) = \psi_0(x, y),$$

Boundary Condition:

$\psi(x, y, t) = 0$ on the boundary of $(0, \pi)\times(0, \pi)$

Galerkin Method:

1. Choose a basis $\phi_{ij}(x, y)$ for $L^2((0, \pi) \times (0, \pi))$ which satisfy the boundary condition (say $\phi_{ij} = \sin ix \sin jy$ or $\phi_{ij} = T_i(x)T_j(y)$ where T_i is related to the Chebyshev polynomial on $(0, \pi)$).

2. Seek a solution of the form
$$\psi = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(t)\phi_{ij}(x, y).$$

3. Substitute ψ into PDE, take inner product with ϕ_{mn}, $m = 1, 2, ..., N$, $n = 1, 2, ..., N$ and end up with N^2 **Ordinary** differential equations for a_{ij}. Solve this initial-value problem to get an approximate solution.

Let

$$L[\psi] = \Delta\psi_t + RJ(\psi, \Delta\psi) + \psi_x + \epsilon\Delta\psi + F$$

Galerkin method reduces to:

$$\int\int_D L[\psi]\phi_{mn}\, dxdy = 0,$$

$$m = 1, ..., N, \quad n = 1, ..., N,$$

where

$$\psi = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t)\phi_{ij}(x, y).$$

Computational Symbolic Algebra reduces the problem to N^2 ODEs in N^2 unknowns.

Case $N = 2$: Let

$$\phi_{ij} = \sin ix \sin jy.$$

Then

$$\psi = a(t) \sin x \sin y + b(t) \sin 2x \sin y +$$

$$c(t) \sin x \sin 2y + d(t) \sin 2x \sin 2y$$

a, b, c, d satisfy the system:

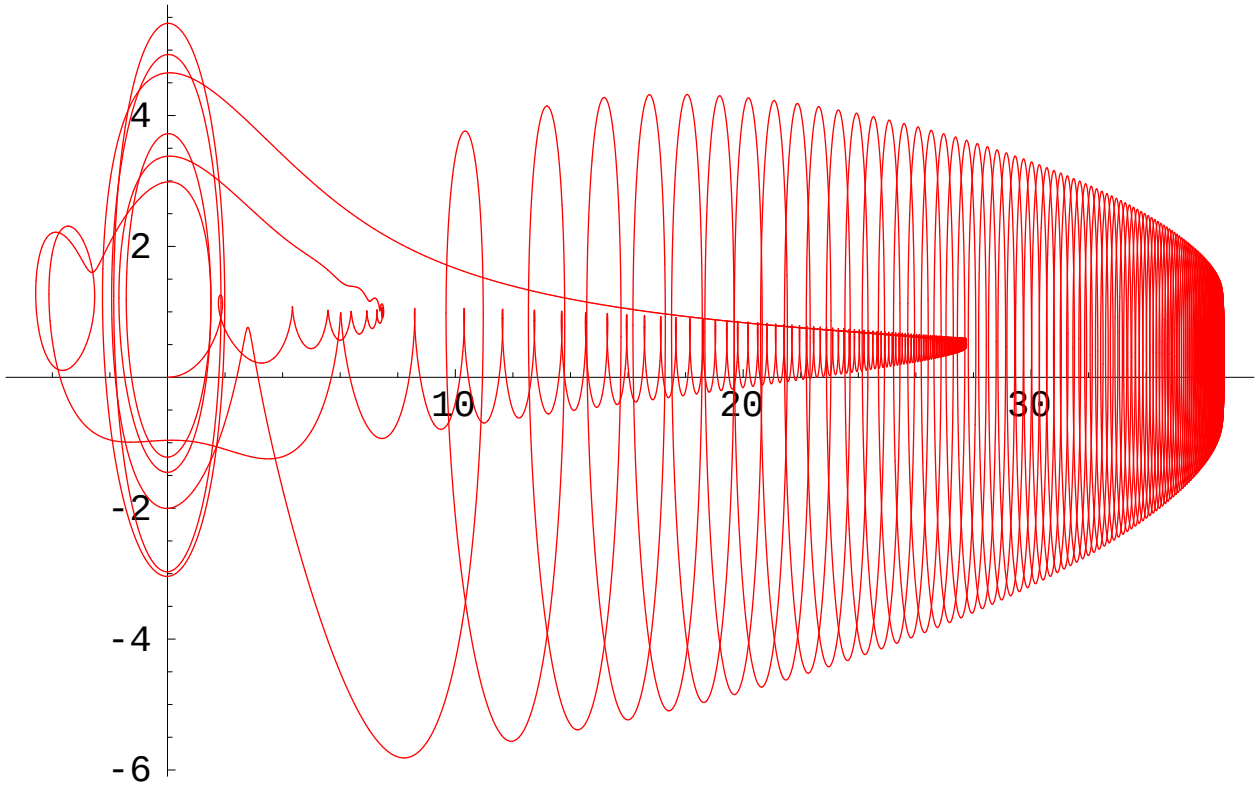
$$a' = -\frac{4}{3\pi} b - \epsilon a + \frac{1}{2},$$

$$b' = \frac{8}{15\pi} a + \frac{9}{20} Rac - \epsilon b,$$

$$c' = -\frac{8}{15\pi} d - \frac{9}{20} Rab - \epsilon c,$$

$$d' = \frac{1}{3\pi} c - \epsilon d.$$

First Two Coefficients, R = 0.3, eps =0.01, N = 2

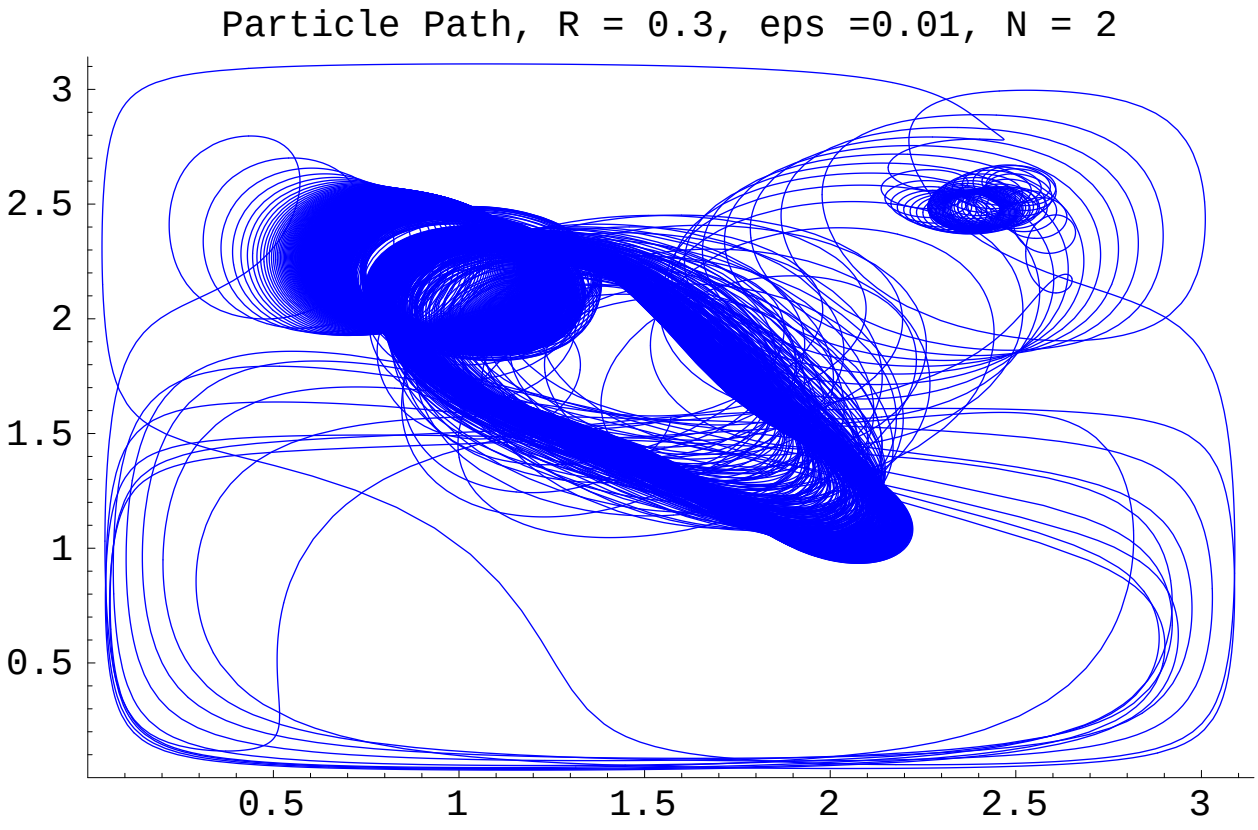


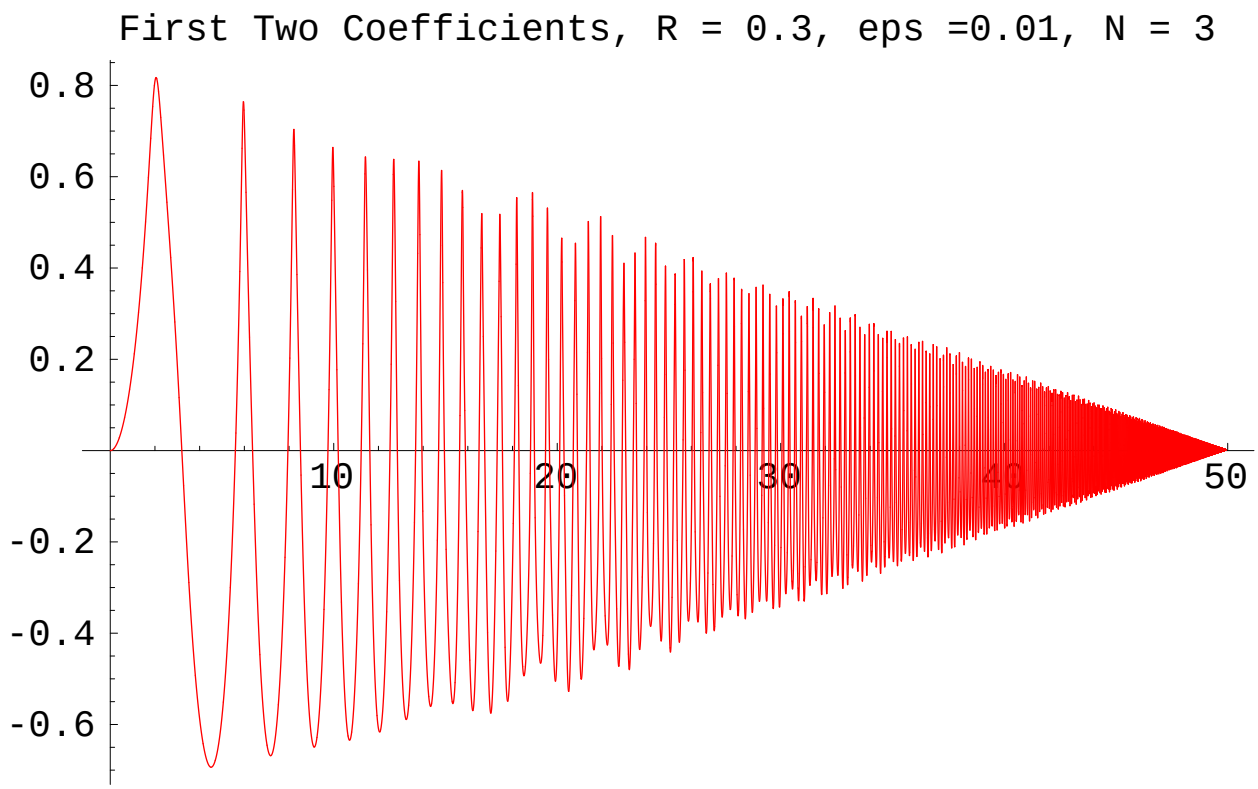
Dynamical System – (Particle Paths)

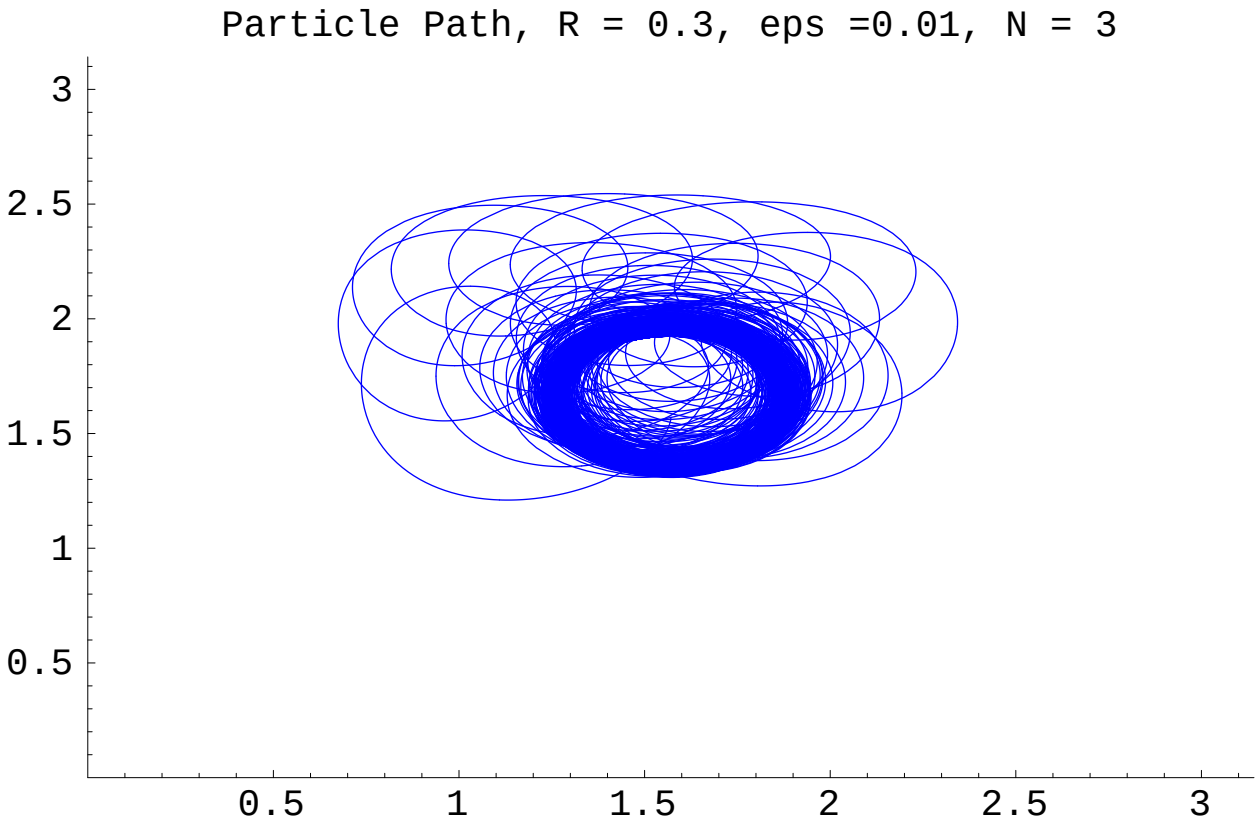
$$\frac{dx}{dt} = -\frac{\partial \psi}{\partial y}, \qquad \frac{dy}{dt} = \frac{\partial \psi}{\partial x}$$

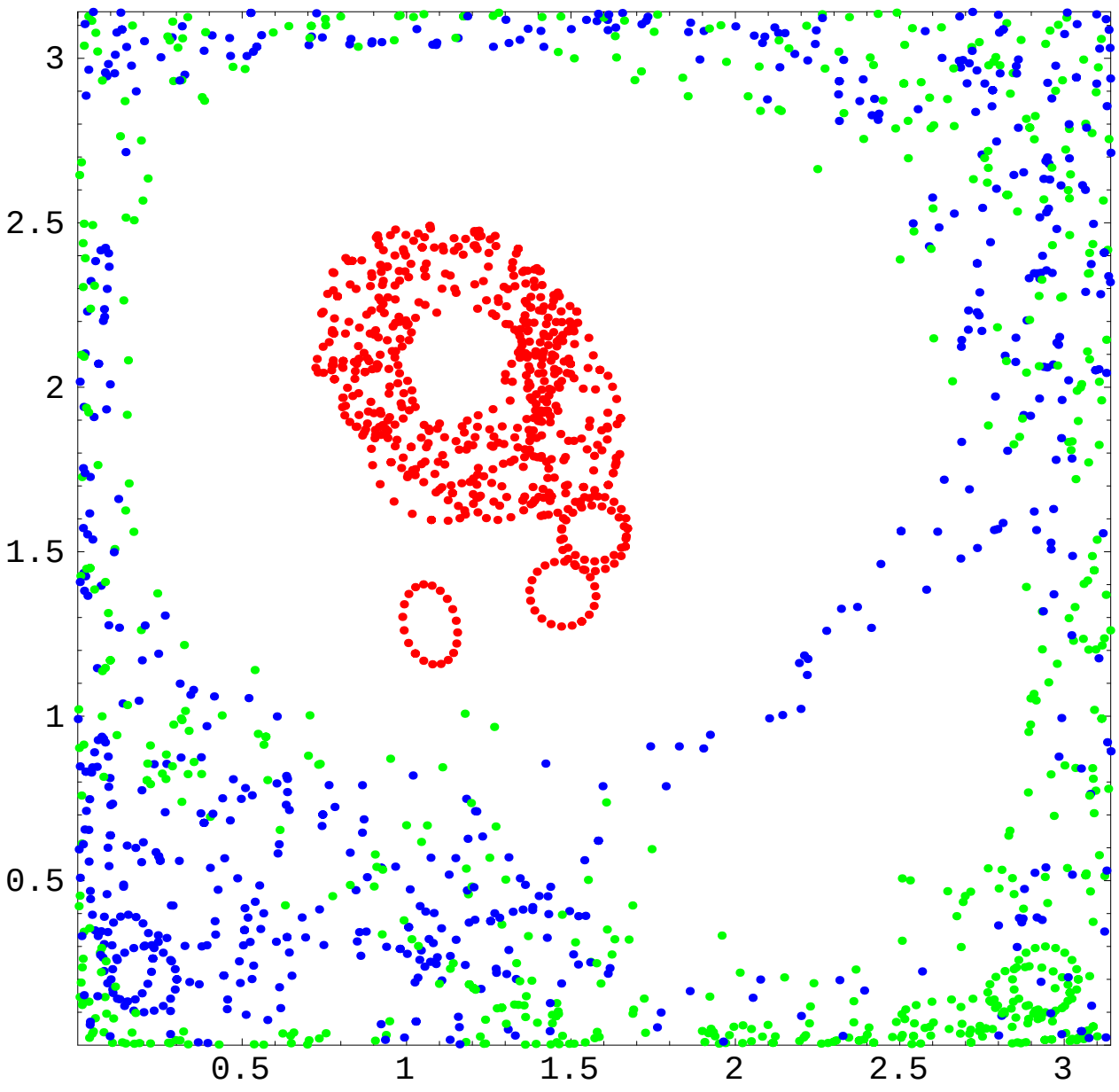
where

$$\psi(x, y, t) = \sum_{n=1}^{N} \sum_{n=1}^{N} a_{ij}(t)\phi_{ij}(x, y).$$


Particle Path, R = 0.3, eps =0.01, N = 2
3
2.5
2
1.5
1
0.5
0.5
1
1.5
2
2.5
3


First Two Coefficients, R = 0.3, eps =0.01, N = 3
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
10
20
30
40
50


Particle Path, R = 0.3, eps =0.01, N = 3
3
2.5
2
1.5
1
0.5
0.5
1
1.5
2
2.5
3


3
2.5
2
1.5
1
0.5
0.5
1
1.5
2
2.5
3

Lorenz's Initial-Boundary Value Problem

Boussinesq Approximation:

$$\rho = \rho_0(1 - \alpha(\theta - \theta_0))$$

Models a 2-D atmosphere, heated from the bottom. When δt is "small", the equilibrium solution is stable. At a critical value of δt (Rayleigh Number), the equilibrium solution becomes unstable and rolls develop. Lorenz's model is intended for this regime.

Equations (ψ = Stream function, θ = temperature, ν = viscosity, κ = thermal diffusivity, α = volume expansion)

$$\Delta\psi_t + J(\psi, \Delta\psi) = \nu\Delta^2\psi + g\alpha\frac{\partial\theta}{\partial x}$$

$$\Delta\theta_t + J(\psi, \theta) = \nu\Delta\theta + \frac{\delta t}{H}\frac{\partial\psi}{\partial x}.$$

Domain $D = \{(x,z)|\, 0 < x < \frac{H}{a}, 0 < z < H\}$

Boundary Conditions: $\psi =$ zero everywhere on ∂D and θ zero at $z = 0$ and $z = H$ and insulated at $x = 0$ and $x = \frac{H}{a}$.

Prandtl Number $P = \frac{\nu}{\kappa}$ (approximately 10)

Rayleigh Number $R_a = \frac{g\alpha H^3 \delta t}{\nu\kappa}$

Critical Number $R_c = \frac{\pi^4(1+a^2)^3}{a^2}$ (minimum occurs when $a^2 = \frac{1}{2}$)

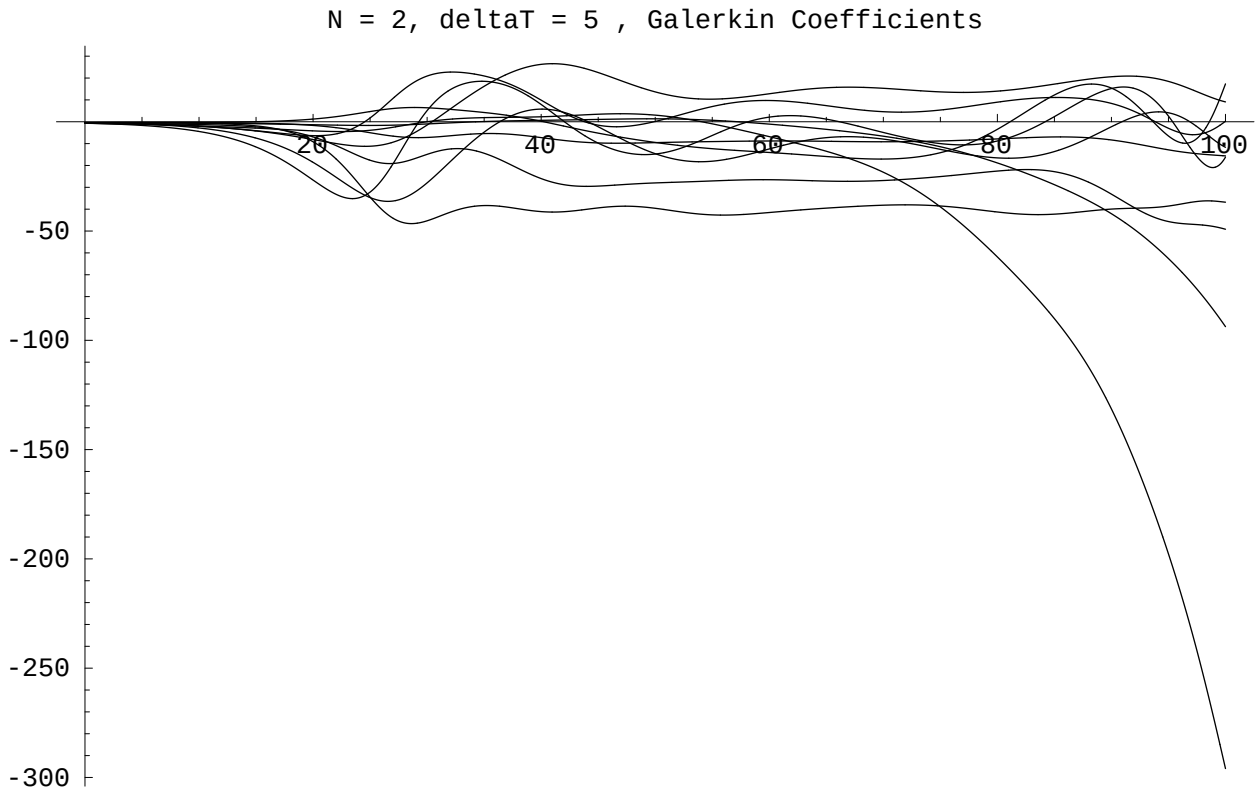
Condition for existence of rolls $R_a > R_c$

Lorenz's condition: $r = \frac{R_a}{R_c} = 28$

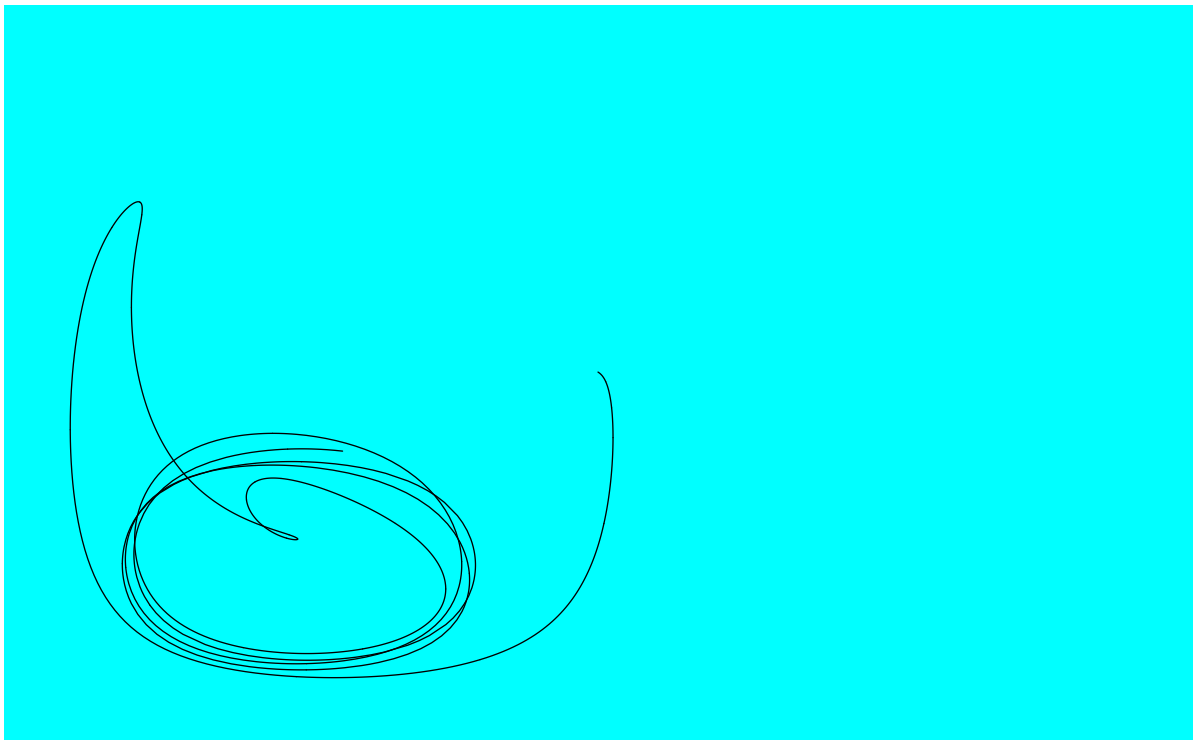
Galerkin Method:

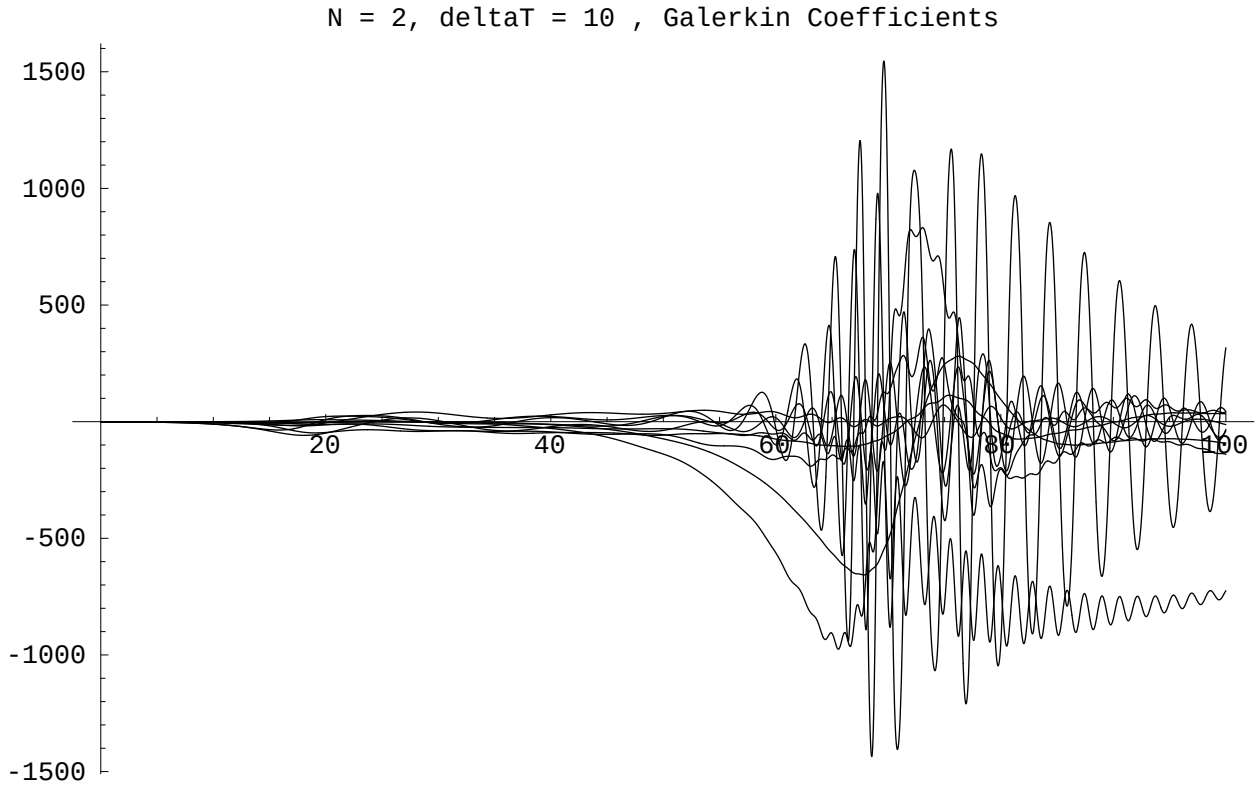
$$\psi(x,z,t) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(t) \sin \frac{ia\pi x}{H} \sin \frac{j\pi z}{H},$$

$$\theta(x,z,t) = \sum_{i=0}^{N} \sum_{j=1}^{N} b_{ij}(t) \cos \frac{ia\pi x}{H} \sin \frac{j\pi z}{H},$$

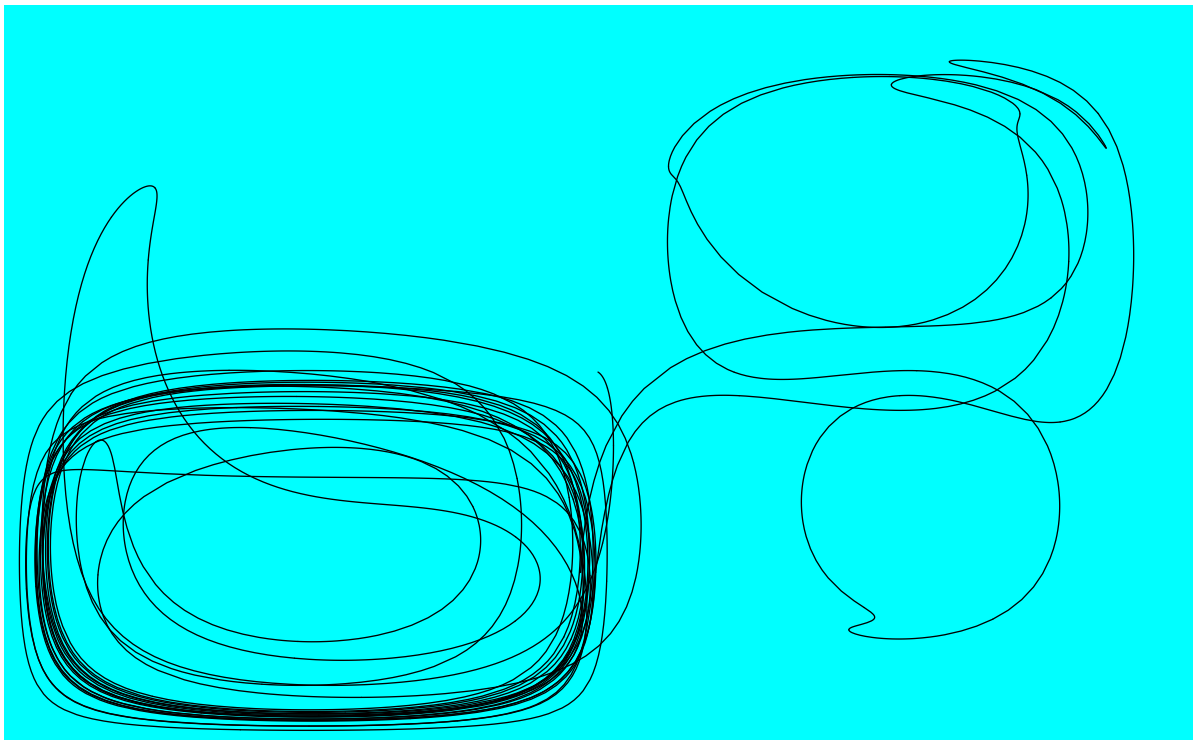

N = 2, deltaT = 5 , Galerkin Coefficients
20
40
60
80
100
-50
-100
-150
-200
-250
-300

N = 2 ,deltaT = 5 , Fluid Flow




N = 2, deltaT = 10 , Galerkin Coefficients
1500
1000
500
-500
-1000
-1500
20
40
60
80
100

N = 2 ,deltaT = 10 , Fluid Flow



1. When is the hamiltonian system asymptotically autonomous?

2. When is the hamiltonian system asymptotically periodic? (Dafermos, 1972)

3. If not asymptotically periodic, then what? (Wiggins, DHT, 2000)

4. What is the impact of choice of bases on complexity?